The two-dimensional spin-1 Ising system and related models

1984 J. Phys. A: Math. Gen. 17 2233

(http://iopscience.iop.org/0305-4470/17/11/019)

J. Phys. A: Math. Gen. 17 (1984) 2233–2345. Printed in Great Britain

The two-dimensional spin-1 Ising system and related models

Joan Adler† and I G Enting‡

† Solid State Institute and Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel

‡ CSIRO, Division of Atmospheric Research, Private Bag No 1, Mordialloc, Victoria, Australia, 3195

Received 7 July 1983, in final form 1 March 1984

Abstract. Although the two-dimensional spin-$\frac{1}{2}$ Ising model was solved in zero field in 1944, no exact results are yet available for the spin-1 model. This model should have the same dominant critical exponents as the spin-$\frac{1}{2}$ model, but appears to exhibit non-analytic corrections to scaling as expected for a ϕ^4 model and unlike the spin-$\frac{1}{2}$ model. In this paper new 45-term low temperature series for the magnetisation, specific heat and susceptibility of the spin-1 model on the square lattice are presented and analysed. A new 24-term staggered susceptibility series for the hard square model is presented and the extant order parameter series for this model (which is also in the ϕ^4 universality class) are also considered. For both models a non-analytic confluent correction with exponent $1.0 < \Delta_1 < 1.3$ is found. The validity of this result is enhanced by a comparison with the $S = \frac{1}{2}$ case.

1. Introduction

One of the anomalies in the area of phase transitions that has led to considerable complications is the absence of non-analytic corrections in the exactly solved spin-$\frac{1}{2}$ Ising model. These corrections are predicted by renormalisation group (RG) theory (Wegner 1972) and are present in 3D Ising models (Chen *et al* 1982, Adler *et al* 1982b and references therein, Adler 1983a) and many other 2D systems (for example the exactly solved Baxter–Wu model (Joyce 1975, Adler 1983b), percolation (Adler *et al* 1982a, 1983b) and the three-state Potts model (Adler and Privman 1982, 1983a)). Neglect of their presence in 3D Ising systems led to apparent violations of hyperscaling; however, it is now established that they are present, not only in the continuous spin model, but also for all spin values in 3D except perhaps near one particular S value, where the amplitude of the non-analytic correction may vanish (Chen *et al* 1982). It is the aim of the present paper to establish their presence for $S = 1$ in 2D, in agreement with RG predictions for a ϕ^4 system. We will also demonstrate their existence in another 2D ϕ^4 spin system, the hard square model (Baxter *et al* 1980 and references therein). The method of analysis used in both cases is that developed by Adler *et al* (1983b) from the transformation of Roskies (1981). For the $S = 1$ Ising model similar results are found with another method (Adler *et al* 1981).

In the exact solution for the 2D spin-$\frac{1}{2}$ Ising model analytic corrections to scaling are present and have recently been shown (Aharony and Fisher 1980, 1983) to arise via nonlinear scaling fields. These are probably also present in the spin-1 model, but should in no way preclude the observation of non-analytic corrections. The analytic correction to the (2D spin-$\frac{1}{2}$) susceptibility and magnetisation can be clearly observed,

0305-4470/84/112233 + 13$02.25

using the method of Adler *et al* (1983b); the non-analytic corrections obtained in the present work take a quite different form.

In § 2 we present extended low temperature series for the $S=1$ Ising model on the square lattice and a new series for the hard square model. Two of the low temperature series are analysed together with an extant hard square series in § 3 to give estimates of T_c, γ, β and Δ_1. Some new results for spin-$\frac{1}{2}$ are presented in § 4 and discussion of all the results and a comparison with Δ_1 estimates from other methods given in § 4. The new series are presented in the appendix. We note that the $S=1$ Ising and hard square models are two of the simplest ways of generalising the $S=\frac{1}{2}$ Ising model to search for non-analytic corrections to scaling.

2. Derivation of the new series

The low temperature series expansions for the spin-1 Ising model that we consider are essentially the same as those investigated by Fox and Guttmann (1973). We have extended the series from order u^{26} to u^{45} and have corrected a few minor errors. The main difference is that we begin by considering the partition function rather than the free energy.

We write the Hamiltonian for the spin-1 Ising model as

$$\mathscr{H}=\sum_{\langle ij\rangle} J(1-S_iS_j)+\sum_i H(1-S_i) \tag{2.1}$$

where as usual the first sum is over all bonds on the square lattice and the second sum is over all sites. The constants are included so that the ground state will have zero energy. This removes awkward constants from the finite lattice formalism without affecting the series. The spin variables S_i can take values 1, 0, −1.

The low-temperature/high-field expansion has been described by Sykes and Gaunt (1973). It is based on perturbations about the $S_i=1$ ground state and leads to a double power series in the variables

$$u=\exp(-J/kT), \qquad \mu=\exp(-H/kT). \tag{2.2a, b}$$

If we concern ourselves with the temperature grouping i.e. the expansion in powers of u, then we have

$$Z=1+\sum_{n=4}^{\infty} u^n\Psi_n(\mu)=1+u^4\mu+2u^7\mu^2+\dots \tag{2.3}$$

where $\Psi_n(\mu)$ are polynomials in μ. It is possible to re-express these polynomials as polynomials in $x=1-\mu$ and to expand the partition function as

$$Z=Z_0(u)+xZ_1(u)+x^2Z_2(u)+\dots. \tag{2.4}$$

At zero field, $x=0$ and we can define the free energy

$$F=-kT\ln Z_0(u), \tag{2.5}$$

the spontaneous magnetisation

$$M(u)=M(0)+(\partial/\partial H)\ln Z=1-Z_1(u)/Z_0(u) \tag{2.6}$$

and the initial susceptibility

$$\chi=kT[2Z_2(u)/Z_0(u)-Z_1(u)/Z_0(u)-(Z_1(u)/Z_0(u))^2]. \tag{2.7}$$

These thermodynamic functions can be obtained with expansion (2.4) truncated at order x^2. In the finite lattice calculations we can work in terms of u and x and truncate all intermediate expressions at order x^2, giving a considerable reduction in the amount of computation required, compared with working with μ.

The following description of the finite lattice technique follows the formalism described by Enting (1978a), except for the use of one-site-at-a-time transfer matrices.

We calculated finite lattice partition functions Z_{mn} for rectangles of n sites by m sites, surrounded by a boundary of $2(n+m)$ sites whose spins were fixed into state 1. Thus

$$Z_{nm} = \sum_{\text{spin states}} \exp\left(-\beta J \sum_{\langle ij\rangle} (1 - S_i S_j) - \beta H \sum_i (1 - S_i)\right) \tag{2.8}$$

where

(i) the sum over spin states is over all 3^{nm} states of the spins in the rectangle,

(ii) the sum over bonds is over all $2nm + m + n$ bonds that connect a spin in the rectangle to another such spin *or* to one of the boundary spins,

(iii) the sum over spins is over all nm spins in the rectangle.

It is widely known that we can obtain series expansions from the approximation

$$Z \approx \prod_{m,n} Z_{mn}^{a_{mn}}. \tag{2.9}$$

Relation (2.8) is possibly most familiar in the form obtained by taking the logarithms of each side: the free energy is a linear combination of finite lattice free energies. The combinatorial ideas go back at least to the work of Hijmans and de Boer (1955). Enting (1978b) gave explicit expressions for the a_{mn} in many cases.

For the spin-1 Ising model (and most other temperature grouping problems) it is most efficient to use

$$\begin{aligned} a_{mn} &= 1 && \text{if } m+n = 2\omega_{\max}+1 \\ &= -3 && \text{if } m+n = 2\omega_{\max} \\ &= 3 && \text{if } m+n = 2\omega_{\max}-1 \\ &= -1 && \text{if } m+n = 2\omega_{\max}-2 \\ &= 0 && \text{otherwise} \end{aligned} \tag{2.10}$$

where $\omega_{\max}$ is the largest width for which we can calculate Z_{mn} (exploiting the $Z_{mn} = Z_{nm}$ symmetry).

The number of terms given correctly by (2.9) is determined by the power of the lowest-order connected graph that does not fit into any of the rectangles considered. With the 'cut-off' given by (2.10) the pertinent graphs are chains of $2\omega_{\max}+1 = r$ sites all in the '0' state. These have power u^{3r+1} and so the series will be correct to order $u^{3r} = u^{6\omega_{\max}+3}$. We have used $\omega_{\max} \approx 7$ which means that our series should be correct through u^{45}. We have explicitly checked the predicted form of the cut-off by repeating our calculations using $\omega = 1, 2, 3, 4, 5$ and 6 noting that only terms to $u^{6\omega+3}$ agree with our final series.

The real power of the finite lattice method comes from the fact that the Z_{mn} can be easily calculated using transfer matrix techniques. The most efficient form seems to be to use a formalism that adds one site at a time rather than one row at a time. This type of transfer matrix has been described by Enting (1980) in connection with polygon enumerations. As well as speeding up the computation, the use of these transfer matrices simplifies the procedures for calculating transfer matrix elements so that we can avoid having to store a set of 2187 × 2187 transfer matrix elements. (Enting (1978a) pointed out several special cases in which matrix elements of conventional row-at-a-time transfer matrices can be easily obtained. The site-at-a-time transfer matrices make this trick feasible for a wider class of interactions.) The calculations were performed using residue arithmetic modulo five different prime numbers. This means that the series are correct modulo $X = \Pi_{i=1}^{5}(2^{15} - a_i)$, with $a_i = 19$, 49, 51, 55 and 61.

The final terms (u^{25} and u^{26}) given by Fox and Guttmann (1973) disagree with the series that we have given in the appendix. A comparison indicates that Fox and Guttmann have most probably omitted a contribution of $4\mu^{14}u^{25}$. We are not able to deduce a probable form of the μ-dependent correction at order u^{26} because the discrepancy appears to involve several powers of μ.

The degree of automation in the spin-1 series calculations together with the consistency checks involved in using various widths lead us to believe that our series are correct.

For the hard square model, series for the order parameter (i.e. the staggered density) were taken from Baxter *et al* (1980). They did not obtain series for the staggered susceptibility because no staggered field was included in their calculations. They were also able to obtain series for the staggered density by making use of the distinction between the sublattices. In the appendix we present shorter series for the low-density staggered susceptibility. These have been calculated using the techniques described by Enting (1978a, b), and are an extension of the series given by Gaunt and Fisher (1965).

3. Results for T_c, γ, β and Δ_1

The series for $M(u)$, $\chi(u)$ and $C_v(u)$ for the $S = 1$ model are assumed to have critical behaviour of the forms

$$M(u) \sim (u_c - u)^{\beta}[1 + a_{1M}(u_c - u)^{\Delta_1} + b_{1M}(u_c - u) + \ldots], \tag{3.1}$$

$$\chi(u) \sim (u_c - u)^{-\gamma}[1 + a_{1\chi}(u_c - u)^{\Delta_1} + b_{1\chi}(u_c - u) + \ldots], \tag{3.2}$$

$$C_v(u) \sim (u_c - u)^{-\alpha}[1 + a_{1C}(u_c - u)^{\Delta_1} + b_{1C}(u_c - u) + \ldots]. \tag{3.3}$$

Although in the case of the $S = \frac{1}{2}$ model the a_1 are zero, the exponents in the $S = 1$ case are expected (by universality) to take the $S = \frac{1}{2}$ values $\beta = \frac{1}{12}$, $\gamma = \frac{7}{4}$ and $\alpha = 0$. We have analysed these series with the usual Dlog Padé approximant technique; some selected approximants are presented in table 1 and for purposes of comparison approximants for the $S = \frac{1}{2}$ magnetisation are also given. We see very consistent behaviour in the spin-$\frac{1}{2}$ model but the spin-1 results are less internally consistent and furthermore the exponent results for the spin-1 model are not in complete agreement with the exact results. With the exception of the C_v results typical Padés are presented. In both the spin-1 and spin-$\frac{1}{2}$ magnetisation series a very few Padés (for example the [23, 21] and [20, 19] Padés respectively) have residues quite different to the majority. This may be related to a phenomenon that is discussed in § 4 below. Analysis with the usual Dlog Padé approximant technique is equivalent to assuming $a_1 = 0$ (Adler *et al* 1982a). However, for the $S = 1$ model we suspect $a_1 \neq 0$ since this is the RG

Table 1. (*a*) Estimates for u_c and β from the $[L, M]$ Padé approximant to the $M(u)$ series for $S = 1$.

N						
44	$[L, M]$	[20, 24]	[21, 23]	[22, 22]	[23, 21]	[24, 20]
	u_c	0.554 128	0.554 148	0.554 126	0.554 139	0.554 138
	β	0.126 040	0.126 260	0.126 019	0.025 961 4	0.126 15
43	$[L, M]$	[20, 23]	[21, 22]	[22, 21]	[23, 20]	
	u_c	0·554 154	0.554 156	0.554 137	0.554 137	
	β	0.114 697	0.122 226	0.133 767	0.127 869	
42	$[L, M]$	[20, 22]	[21, 21]	[22, 20]	[23, 19]	
	u_c	0.554 145	0.554 284	0.554 139	0.554 138	
	β	0.126 228	0.127 593	0.139 675	0.126 135	
41	$[L, M]$	[19, 22]	[20, 21]	[21, 20]	[22, 19]	
	u_c	0.554 252	0.554 053	0.553 984	0.554 133	
	β	0.118 186	0.125 255	0.124 482	0.130 351	

Table 1. (*b*) Estimates for u_c and γ from the $[L, M]$ Padé approximant to the $\chi(u)$ series.

N						
40	$[L, M]$	[18, 22]	[19, 21]	[20, 20]	[21, 19]	[22, 18]
	u_c	0.554 432	0.554 431	0.554 929	0.554 847	0.554 647
	γ	1.825 31	1.825 18	1.892 76	1.901 04	1.852 82
39	$[L, M]$	[18, 21]	[19, 20]	[20, 19]	[22, 17]	
	u_c	0.554 410	0.555 036	0.554 996	0.554 451	
	γ	1.822 65	1.908 54	1.723 55	1.827 74	
38	$[L, M]$	[16, 22]	[18, 20]	[19, 19]	[20, 18]	[21, 17]
	u_c	0.553 675	0.556 397	0.554 892	0.554 274	0.554 089
	γ	1.747 86	2.176 53	1.258 51	1.294 1	1.785 75
37	$[L, M]$	[17, 20]	[18, 19]	[19, 18]	[21, 16]	
	u_c	0.553 197	0.554 586	0.553 233 0	0.553 443	
	γ	1.712 80	1.846 82	1.699 57	1.718 34	

Table 1. (*c*) Estimates for u_c and α from the $[L, M]$ Padé approximant to the $C_v(u)$ series.

$[L, M]$	[15, 25]	[15, 24]	[14, 24]
u_c	0.523 82	0.525 201	0.523 832
α	0.030 390 6	0.019 843 8	0.038 706 5
$[L, M]$	[13, 24]	[16, 21]	[12, 24]
u_c	0.522 501	0.511 163	0.522 533
α	0.061 298 6	0.006 293 61	0.002 869 3

Table 1. (*d*) Estimates for u_c and β from the $[L, M]$ Padé approximant to the $M(u)$ series for $S=\frac{1}{2}$.

N				
			20 term series	
19	$[L, M]$	[9, 10]	[10, 9]	
	u_c	0.171 572	0.171 572	
	β	0.124 999	0.124 999	
18	$[L, M]$	[8, 10]	[9, 9]	[10, 8]
	u_c	0.171 572	0.171 572	0.171 572
	β	0.124 999	0.125 000	0.124 999
			40 term series	
39	$[L, M]$	[19, 20]	[20, 19]	[21, 18]
	u_c	0.171 572	0.171 572	0.171 572
	β	0.125 000	0.000 969	0.125 000
38	$[L, M]$	[18, 20]	[19, 19]	[20, 18]
	u_c	0.171 572	0.171 572	0.171 572
	β	0.125 000	0.125 000	0.125 000

prediction and we expect that the $D=2$ spin-$\frac{1}{2}$ Ising model is a very special exception to this general behaviour. The predicted values for the exponent Δ_1 are 1.4 (Le Guillou and Zinn-Justin 1980) and 1.4 ± 0.8 (Baker *et al* 1978, Baker 1983). Thus it is interesting to see what the effect of $\Delta_1\neq 1$ is on the estimates of γ, β and u_c.

We have studied this effect and made an independent evaluation of Δ_1 with the methods of Adler *et al* (1983b) and Adler *et al* (1981). The former method involves minimising the effect of the correction on the evaluation of the dominant exponent, and is a generalisation of the transform of Roskies (1981), whereas the latter method gives us a corroborating estimate of Δ_1. In the former method we transform the series $A(u)$ in u to ones in

$$y=1-(1-u/u_c)^{\Delta_1}$$

and then look at different Padé approximants to the function

$$G_\Delta(y)=\Delta(y-1)(\mathrm{d}/\mathrm{d}y)(\ln A(y))=h-p/(1+p)$$

where h is the dominant exponent and $p=a_1u_c^{\Delta_1}\Delta_1(y-1)^{\Delta_1/\Delta}$. The correction term p becomes zero when $u=u_c$ and $\Delta=\Delta_1$. We evaluate the Padé approximants for a range of guesses at u_c and Δ; for the correct Δ_1 and u_c these Padé approximants should intersect and give a correct estimate for the dominant exponent. For a model where neither u_c nor Δ_1 are known we search for the best convergence in the (u_c, h, Δ_1) space (Adler 1983a); here, however, we know the values of the dominant exponents via universality arguments and thus we search for the best Δ_1 and u_c consistent with these. We note that a very strong check of the validity of our results is if the same u_c and Δ_1 values are observed for all quantities studied, and if the best convergence is indeed found for the universal values of the dominant exponents. The latter method is believed (Adler *et al* 1983b) to be most reliable for Δ_1 close to 1.0. It involves studying Padé approximants to the logarithmic derivative of $B(u)$ where

$$B(u) = hA(u) + (u_c - u)\, dA(u)/du.$$

This logarithmic derivative has a pole at u_c with residue $h + \Delta_1$; thus here the input into the calculation is u_c and h. We may again search for intersection regions of the different Padé approximants in the u_c, h, Δ_1 plane and the Δ_1 estimates from this method should be consistent with those of the former one.

We have analysed both the magnetisation and the susceptibility series with the former method and the magnetisation series with the latter method. The Padé results from the specific heat series were not sufficiently defect and problem free to justify further study. The results of the susceptibility analysis are presented in figure 1 and the results of the magnetisation analysis by the former method are presented in figure 2. We find that the best convergence for $\gamma = 1.75$ and $\beta = 0.125$ is found for $u_c \sim 0.554\,06$ in both cases. Furthermore Δ_1 estimates are centred around 1.175 in both cases. We present the alternative magnetisation analysis for $u_c = 0.554\,06$ in figure 3 and here observe that Δ_1 is again close to or slightly above 1.175. By looking at plots for $0.5530 < u_c < 0.5542$ we find the Δ_1 estimate to be

$$1.0 < \Delta_1 < 1.3.$$

In both cases Δ_1 estimates decrease as u_c increases. Allowing for ranges $0.12 < \beta < 0.13$ and $1.7 < \gamma < 1.8$ we find $0.5538 < u_c < 0.5542$.

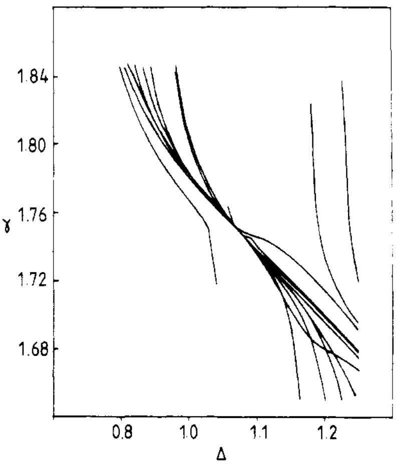

Figure 1. Graph of γ against Δ for the low temperature susceptibility of the spin-1 Ising model at $u_c = 0.554\,06$.

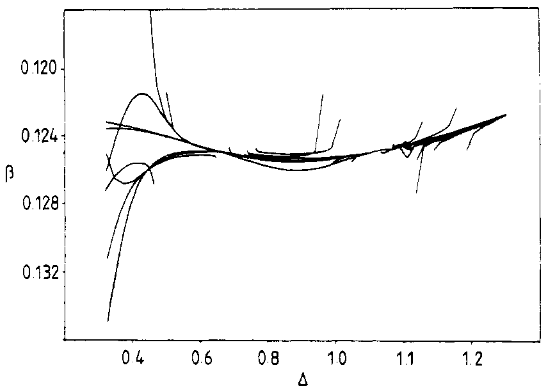

Figure 2. Graph of β against Δ for the magnetisation of the spin-1 Ising model at $u_c = 0.554\,06$, using the method of Adler *et al* (1982a).

We note that in figures 1 and 2 the exponent values near $\Delta_1 = 1.0$ are similar to those obtained in the Padé study (see above), $\gamma \gtrsim 1.76$ and $\beta \lesssim 0.1255$, thus supporting our statement that the Padé results are equivalent to $\Delta_1 = 1.0$.

We now consider the hard square series. The series that we consider in depth is that for the staggered density R (as a function of x, the inverse of the activity) which is the analogue of the magnetisation. If the hard square model indeed falls in the ϕ^4 universality class these quantities should have the same dominant exponent β. The R

series (Baxter *et al* 1980) are 24 terms long and alternate regularly in sign. The results for $\rho_c = 0.263\,413$ are shown in figure 4. We observe that there is an intersection region exactly at $\beta = \frac{1}{8}$ and that $1.2 < \Delta_1 < 1.4$. If we consider the range $0.263\,41 < u_c < 0.263\,42$ we find $1.0 < \Delta_1 < 1.3$. We have also studied R as a function of $\rho' = 1 - 2\rho$, where ρ is the density.

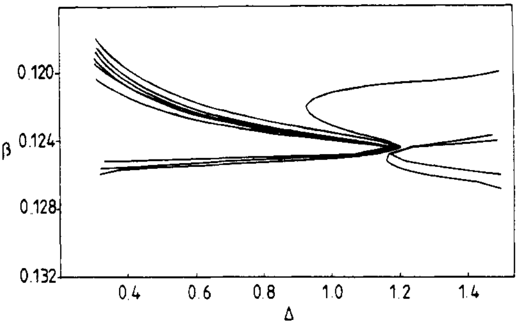

Figure 3. Graph of β against Δ for the magnetisation of the spin-1 Ising model at $u_c = 0.554\,06$, using the method of Adler *et al* (1981).

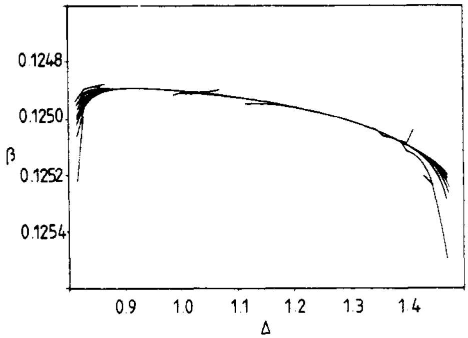

Figure 4. Graph of β against Δ for the hard square staggered density R at $\rho_c = 0.263\,413$.

This quantity should have a critical exponent of $\beta/1-\alpha$; however, Baxter *et al* were unable to obtain the expected exponent of $\frac{1}{8}$ from a Padé analysis. We met with a similar lack of success, although for $\rho' \sim 0.2648$ we obtain clear convergence with $1.0 < \Delta_1 < 1.2$. We do not find clear convergence within the range 0.264 ± 0.002 given by Baxter *et al* although at the bottom of the range (~ 0.2638) the results are not inconsistent with $\beta = \frac{1}{8}$. We were unable to analyse the staggered susceptibility series with Padé methods.

4. New results for the spin-$\frac{1}{2}$ Ising model

In this section we present some previously unpublished results for the spin-$\frac{1}{2}$ Ising model; these will be used as a basis for comparison with the above.

In figure 5 we show a new analysis of Nickel's (1982) 34-term high temperature susceptibility series. Here we have a clear case of an analytic correction, and indeed excellent convergence is observed for $\gamma = 1.75$ and $\Delta_1 = 1.0$ We also have convergence at $\gamma = 1.75$ and $\Delta_1 = 0.5$ and again near $\gamma \sim 1.75$ and $\Delta_1 \sim 0.33$. The convergences at $\gamma \sim 1.75$ for $\Delta_1 = 0.5$ and 0.33 appear to be 'resonances'. These have been previously obtained by Privman (1983), who studied test series with the method of Adler *et al* (1983b), but do not appear to have been seen previously in studies of 'real' systems. Privman (1983) explains that these 'resonances' at values of $\Delta_1 = \Delta_1/k$, where $k = 2, 3, \ldots$, are to be expected in this method; the surprising fact is that they were not previously observed. They have recently also been observed in the specific heat series for the Baxter–Wu model (Adler 1983b). Both these quantities are exactly solvable and apparently lack the type of terms that destroy (Privman 1983) the convergence regions for $k > 1$. We note that all these 'resonances' fall at the same value of γ as the main convergence region near $\Delta_1 = 1$; and from the ratio of Δ_1 values it is thus

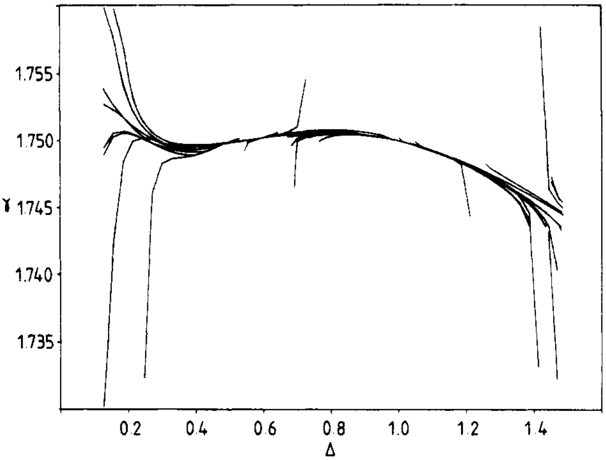

Figure 5. Graph of γ against Δ for the high temperature susceptibility of the spin-$\frac{1}{2}$ Ising model at the critical temperature. The series is the 32-term series of Nickel (1982).

easy to see which is the correct Δ_1 value. The 'resonance' near $\Delta = 0.5$ explains the observations of Roskies (1981), who found that applying the transformation

$$y = 1 - (1 - u/u_c)^{1/2}$$

gave the correct values of u_c and γ for the spin-$\frac{1}{2}$ model in 2D, as well as giving the RG values in 3D. One can now observe that this was a very fortunate coincidence, since Roskies' result implied $\Delta_1 = 0.5$ in 2D which is certainly not the case.

In figure 6 we show the $S = \frac{1}{2}$ magnetisation curve for a series of 20 terms. There is an intersection region near $\Delta_1 \sim 1.0$ with 'resonances' near $\Delta_1 \sim 0.1$ and 0.3. The reason that we chose to display the 20-term series is rather interesting. If we consider the highest central Padés in tables for successively longer series the results usually converge as the length of the series increases. Ratio analysis also becomes more

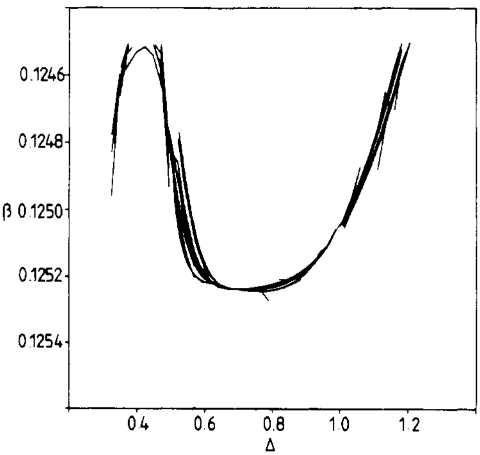

Figure 6. Graph of β against Δ for the magnetisation of the spin-$\frac{1}{2}$ Ising model at the exact u_c. We have obtained this 20-term series by expansion of the exact magnetisation.

convergent as the series becomes longer. For Padé analysis this appears, however, only to be true up to a certain point. When working with very long (say, 40-term) series we do not always observe consistent results for high central approximants; two examples have been given in § 3. This may be due to problems of machine accuracy (although we used 32-figure accuracy throughout our analysis). A similar problem occurs when we take Padé approximants to the function $G_\Delta(y)$. If we look at any of the figures 1, 2, 4 or 5 we see that certain Padés deviate suddenly from the general area and then return. For example, in figure 2 one Padé follows a path with β well below 0.124 near $\Delta_1 = 1$. Thus at $\Delta_1 = 1$ this Padé would have a residue $\ll 0.124$ in disagreement with the others. This phenomenon usually occurs quite rarely but its incidence increases as the length of series increases. The nature and location of the intersection region may improve at the same time. For series that are expansions of exact solutions (such as the spin-$\frac{1}{2}$ Ising or Baxter–Wu magnetisation) this phenomenon seems to occur for relatively few terms in the series, although the location of the intersection region does not seem to move (Adler 1983b). These deviations make the graphs rather confused and thus we present the 20-term graph and in figure 7 we show the spin-1 magnetisation for the 20-term series for comparison. We see that the curves are very different and there are no 'resonances' for Δ/k with $k > 1$. For the 40-term spin-1 series (figure 2) we do see a 'resonance' at 0.6 and this is to date the only non-exactly solved model where this phenomenon has been observed.

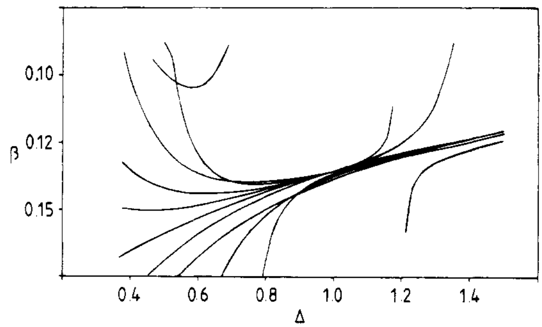

Figure 7. Graph of β against Δ for the magnetisation of the spin-1 Ising model at $u_c = 0.554\,06$. This graph is for a 20-term series and comparison with figure 6 shows that the spin-1 and spin-$\frac{1}{2}$ magnetisations are quite different.

5. Discussion

In the preceding sections we have investigated the critical behaviour of the spin-1 and spin-$\frac{1}{2}$ Ising models and the hard square model, all of which are supposed to have the same dominant exponents. We have presented new series for the spin-1 and hard square models, and it is to be hoped that other methods of series analysis will be applied to these series in the near future.

We have shown that the spin-1 and hard square models exhibit critical behaviour of the form of equations (3.1) and (3.2) with a_{1M} and $a_{1\chi} \neq 0$ and $1.0 < \Delta_1 < 1.3$, whereas it is known for the spin-$\frac{1}{2}$ model that a_{1M} and $a_{1\chi} = 0$, and only b_{1M} and $b_{1\chi} \neq 0$. The results of § 4 demonstrate that our techniques are well capable of providing an accurate description of the analytic correction terms that occur in the spin-$\frac{1}{2}$ model and thus can distinguish between the two possibilities.

There is a constant danger in this kind of work that higher correction terms influence the value of the first correction term and that the $\Delta_1 \neq 1$ we claim to identify is in fact an analytic term. We feel that we can exclude this possibility on the basis that the spin-$\frac{1}{2}$ curves give a clear analytic ($\Delta = 1$) term and, furthermore, the ϕ^4 estimate (see § 3) is $\Delta_1 \sim 1.4$. Since we do not know u_c exactly we cannot prove this beyond all shadow of doubt, nor can we exclude the kind of behaviour recently envisaged (Adler 1983c) for the self-avoiding walk on the honeycomb lattice. This latter scenario finds both an intersection at $\Delta \sim 1.2$ and an intersection near either $\Delta = 1$ or $\Delta < 1$, whereas the field theoretic result is $\Delta_1 \sim 1.2$. However, the strong agreement between the different methods of analysis and the comparison with $S = \frac{1}{2}$ suggests that in this case we do have $\Delta_1 > 1$ for the spin-1 and hard square models. We suggest $1.0 < \Delta_1 < 1.3$.

A new u_c value for the spin-1 Ising model on the square lattice is also proposed. We suggest $0.5538 < u_c < 0.5542$ with a central value of $u_c = 0.554\,06$ on the basis that the best agreement between the dominant exponents of the spin-1 model and the spin-$\frac{1}{2}$ value is found for this value. This u_c value replaces the $u_c = 0.5533 \pm 0.0012$ from Fox and Guttmann (1973).

Acknowledgments

This work was supported in part through NSF grant DMR 78-18808. One of us (JA) acknowledges the support of the Lady Davis Fellowship Foundation and of the Centre for Absorption in Science of the Government of Israel and thanks G Baker for the stimulating discussions and incisive criticism that led to the work reported in § 5. We thank M Revzen for comments on the manuscript, and B Nickel for a discussion on Padé analysis.

Appendix

New series for the spin-1 2D Ising magnetisation ($M(u) = \Sigma_n m_n u^n$), susceptibility ($\chi(u) = \Sigma_n x_n u^n$) and specific heat ($C_v = \Sigma_n c_n u^n$).

n	m_n	x_n	c_n
0	1	0	0
1	0	0	0
2	0	0	0
3	0	0	0
4	−1	1	16
5	0	0	0
6	0	0	0
7	−4	8	98
8	3	−6	−96
9	0	0	0
10	−30	90	1 000
11	48	−144	−1 936
12	−52	192	2 064

n	m_n	x_n	c_n
13	−120	480	5 070
14	368	−1 372	−19 012
15	−612	2 676	31 950
16	−254	1 703	9 024
17	2 524	−11 952	−152 014
18	−6 216	33 316	383 616
19	4 040	−18 900	−298 186
20	11 805	−64 201	−832 320
21	−49 400	304 580	3 575 922
22	68 268	−401 068	−5 486 624
23	14 928	−97 928	−1 012 506
24	−332 511	2 390 637	27 088 992
25	734 508	−5 130 048	−65 115 000
26	−568 038	4 264 858	53 200 524
27	−1 641 320	13 518 716	147 217 176
28	6 202 774	−49 117 798	−608 004 040
29	−9 239 676	76 725 752	947 874 280
30	−2 503 162	29 308 994	189 048 900
31	42 749 908	−381 566 684	−4 568 526 730
32	−99 021 392	915 306 452	11 071 969 920
33	72 255 812	−629 297 848	−8 871 938 526
34	215 763 902	−2 149 429 218	−24 714 851 124
35	−846 523 304	8 606 730 256	102 572 776 040
36	1 235 587 854	−12 408 220 218	−158 562 077 760
37	315 695 688	−3 956 969 996	−31 309 254 516
38	−5 897 043 012	65 853 427 044	766 255 508 396
39	13 498 636 700	−149 789 004 280	−1 846 277 129 736
40	−10 063 784 956	110 599 540 765	1 479 447 715 520
41	−30 197 995 484	371 951 421 160	4 133 610 817 968
42	117 108 185 474	−1 416 033 283 010	−17 054 958 273 276
43	−172 710 840 680	2 102 892 657 652	26 339 112 604 404
44	−46 214 867 144	737 547 145 862	5 331 885 548 880
45	824 863 285 280	−10 822 599 389 744	−127 080 932 186 700

New series for the staggered susceptibility ($\chi^+(\rho) = \Sigma_{n=1}^{24} r_n\rho^n$) of the hard square model as a function of density.

n	r_n	n	r_n
1	1	13	1411 328
2	3	14	4184 264
3	12	15	12 325 012
4	44	16	36 138 680
5	152	17	105 508 964
6	504	18	306 540 276
7	1628	19	886 232 460
8	5176	20	2552 826 468
9	16 276	21	7342 034 404
10	50 632	22	21 113 694 620
11	155 552	23	60 683 948 480
12	471 472	24	173 931 633 140

References

Adler J 1983a *J. Phys. A: Math. Gen.* **16** 3585
—— 1983b *Bulletin IPS* **29** 34 and to appear
—— 1983c *J. Phys. A: Math. Gen.* **16** L515
Adler J, Enting I G and Privman V 1983a *J. Phys. A: Math. Gen.* **16** 1967
Adler J, Moshe M and Privman V 1981 *J. Phys. A: Math. Gen.* **14** L363
—— 1982a *Phys. Rev.* B **26** 1411
—— 1982b *Phys. Rev.* B **26** 3958
—— 1983b in *Percolation Structures and Processes* ed G Deutscher, R Zallen and J Adler *Ann. Israel Phys. Soc.* vol 5 (Bristol: Adam Hilger) p 397
Adler J and Privman V 1982 *J. Phys. A: Math. Gen.* **15** L417
Aharony A and Fisher M E 1980 *Phys. Rev. Lett.* **45** 679
—— 1983 *Phys. Rev.* B **27** 4349
Baker G A 1983 *Private communication*
Baker G A, Nickel B G and Meiron D 1978 *Phys. Rev.* B **17** 1365
Baxter R J, Enting I G and Tsang S K 1980 *J. Stat. Phys.* **22** 465
Chen J H, Nickel B and Fisher M E 1982 *Phys. Rev. Lett.* **48** 630
Enting I G 1978a *Aust. J. Phys.* **31** 515
—— 1978b *J. Phys. A: Math. Gen.* **11** 563
—— 1980 *J. Phys. A: Math. Gen.* **13** 3713
Fox P F and Guttmann A J 1973 *J. Phys. C: Solid State Phys.* **6** 913
Gaunt D and Fisher M E 1965 *J. Chem. Phys.* **43** 2840
Hijmans J and de Boer J 1955 *Physica* **21** 471
Joyce G S 1975 *Proc. R. Soc.* A **345** 277
Le Guillou J C and Zinn-Justin J 1980 *Phys. Rev.* B **21** 3976
Nickel B G 1982 in *Phase Transitions, Proc. 1980 Cargese Summer Institute* ed M Levy, J C Le Guillou and J Zinn-Justin (New York: Plenum)
Privman V 1983 *Preprint*
Roskies R 1981 *Phys. Rev.* B **24** 5305
Sykes M F and Gaunt D S 1973 *J. Phys. A: Math., Nucl. Gen.* **6** 643
Wegner F J 1972 *Phys. Rev.* B **5** 4529, **6** 1891